

Detecting genuine multipartite entanglement in steering scenarios

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(Dated: September 29, 2016)

Einstein-Podolsky-Rosen (EPR) steering is a form of quantum nonlocality which is intermediate between entanglement and Bell nonlocality. EPR steering is a resource for quantum key distribution that is device independent on only one side in that it certifies bipartite entanglement when one party's device is not characterized while the other party's device is fully characterized. In this work, we introduce two types of genuine tripartite EPR-steering, and derive two steering inequalities to detect them. In a semi-device-independent scenario where only the dimensions of two parties are assumed, the correlations which violate one of these inequalities also certify genuine tripartite entanglement. It is known that Alice can demonstrate bipartite EPR-steering to Bob if and only if her measurement settings are incompatible. We demonstrate that quantum correlations can also detect tripartite EPR-steering from Alice to Bob and Charlie, even if Charlie's measurement settings are compatible.

PACS numbers: 03.65.Ud, 03.67.Mn, 03.65.Ta

I. INTRODUCTION

Entanglement, Einstein-Podolsky-Rosen (EPR) steering, and Bell nonlocality are three inequivalent forms of nonlocality in quantum physics [1, 2]. The observation of Bell nonlocality [3, 4] implies the presence of entanglement without the need for the detailed characterization of the measured systems as well as the measurement operators. For this reason, Bell nonlocality has been used as a resource for device-independent (DI) quantum information processing [5, 6]. EPR-steering is a weaker form of quantum nonlocality [1, 7] and is witnessed by violation of a steering inequality [8–10]. EPR-steering is a resource for one-side-device-independent (1SDI) quantum key distribution [11]. This follows from the fact that the observation of EPR-steering in bipartite systems certifies entanglement with measurements on one side characterized and the other side uncharacterized.

The observation of Bell nonlocality or EPR-steering also implies the presence of another nonclassical feature, which is incompatibility of measurements [4, 12]. The notion of commutativity does not properly capture the incompatibility of generalized measurements, i.e., positive-operator-valued measurements (POVMs). The notion of joint measurability [13], which is inequivalent to commutativity, is a natural choice to capture the incompatibility of POVMs. Recently, it has been shown that a set of POVMs can be used to demonstrate bipartite EPR-steering if and only if (iff) it is nonjointly measurable [12, 14, 15]. In other words, Alice's measurement settings are incompatible if she can demonstrate EPR-steering to Bob. On the other hand, Alice can “always find a quantum state” to demonstrate EPR-steering to Bob if her measurement settings are incompatible.

In the multipartite scenario, various approaches to DI verification of genuine entanglement are known [16–23]. Violation of the Bell-type inequalities which detect genuine nonlocality [16–19] belongs to one of these approaches. Verification of multipartite entanglement in partially DI scenarios,

where some of the parties' measurements are trusted, has also been characterized [24–27]. In Ref. [25], genuine multipartite forms of EPR-steering have been developed and criteria to detect them have been derived. The violation of these criteria ensures that steering is shared among all subsystems of the multipartite system. The observation of genuine multipartite steering implies the presence of genuine multipartite entanglement in a 1SDI way. In Ref. [27], tripartite steering inequalities have been derived to detect genuine tripartite entanglement in 1SDI scenarios where either one or two parties' measurements are uncharacterized.

In this paper, we consider two types of tripartite steering scenarios where two parties' measurements are fully characterized (see Fig. 1). We derive two inequalities to detect genuine tripartite steering in these two 1SDI scenarios. We argue that the correlations which violate one of these steering inequalities detect genuine entanglement in a semi-DI way [28] as well. That is, when the Hilbert-space dimensions of two subsystems are constrained, the correlations which exhibit genuine tripartite steering also detect genuine entanglement. We demonstrate that quantum correlations can also detect tripartite EPR-steering from Alice to Bob and Charlie, even if Charlie's measurement settings are compatible.

The paper is organized as follows. In Sec. II, we discuss in detail the three notions of bipartite nonlocality. In Sec. III, we introduce two notions of genuine tripartite EPR-steering and we derive the steering inequalities which detect them. In Sec. IV, we illustrate that incompatibility of POVMs is not necessary for detecting tripartite steering. Conclusions are presented in Sec. V.

II. BIPARTITE NONLOCALITY

Consider a bipartite scenario in which two spatially separated parties, Alice and Bob, perform local measurements on a composite quantum system described by a density operator $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The correlation between the outcomes is described by the conditional probability distributions of getting the outcomes a and b given the measurements A_x and B_y

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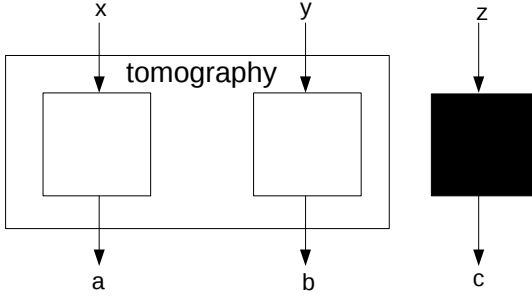


FIG. 1. Depiction of the one-side-device-independent scenario that we consider for deriving the two steering inequalities: Alice and Bob's subsystems and measurements are fully characterized. Charlie's measurements are treated as black box, i.e., Charlie has no knowledge about the measurement operators as well as the measured system.

on Alice and Bob's sides: $P(ab|A_x B_y)$, where x and y label the measurement choices. Quantum theory predicts the correlation as follows:

$$P(ab|A_x B_y) = \text{Tr}(\rho M_{a|x} \otimes M_{b|y}), \quad (1)$$

where $M_{a|x}$ and $M_{b|y}$ are measurement operators.

Bell nonlocality. A correlation is Bell nonlocal if it cannot be explained by the local hidden variable (LHV) model [4],

$$P(ab|A_x B_y) = \sum_{\lambda} p_{\lambda} P_{\lambda}(a|A_x) P_{\lambda}(b|B_y), \quad (2)$$

for some hidden variable λ with probability distribution p_{λ} . In the case of two dichotomic measurements per side, i.e., $x, y \in \{0, 1\}$ and $a, b \in \{-1, +1\}$, the correlation has a LHV model iff it satisfies the Bell–Clauser–Horne–Shimony–Holt (CHSH) inequality [29],

$$\langle A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \rangle_{LHV} \leq 2, \quad (3)$$

and its equivalents [30]. Here, $\langle A_x B_y \rangle = \sum_{ab} ab P(ab|A_x B_y)$. Quantum correlations violate the Bell-CHSH inequality up to the Tsirelson bound $2\sqrt{2}$ [31].

EPR-steering. Consider a 1SDI scenario in which Alice has knowledge about her subsystem and which measurements she can perform, while Bob performs black-box measurements (i.e., uncharacterized measurements). In this scenario, a quantum correlation exhibits EPR-steering from Bob to Alice if it cannot be explained by the hybrid local hidden state (LHS)-LHV model [10],

$$\text{Tr}(\rho M_{a|x} \otimes M_{b|y}) = \sum_{\lambda} p_{\lambda} P(a|A_x, \rho_{\lambda}) P_{\lambda}(b|B_y), \quad (4)$$

where $P(a|A_x, \rho_{\lambda})$ is the distribution arising from quantum state ρ_{λ} . Suppose Alice performs two orthogonal qubit projective measurements; for instance, σ_x and σ_y , and Bob performs two dichotomic black-box measurements. Then the inequality

$$\langle A_0 B_0 - A_1 B_1 \rangle_{2 \times ?}^{LHS} \leq \sqrt{2}, \quad (5)$$

where A_0 and A_1 are qubit projective measurements which satisfy $[A_0, A_1] = -1$, serves as the EPR-steering criterion [8, 9]. Here, $2 \times ?$ indicates that Alice's subsystem is assumed to be qubit while Bob's subsystem is uncharacterized. Just like the Bell-CHSH inequalities, there are eight equivalent steering inequalities. For instance, the steering inequality $\langle A_0 B_1 + A_1 B_0 \rangle_{2 \times ?}^{LHS} \leq \sqrt{2}$ can be obtained from the one in Eq. (5) by the transformation $a \rightarrow a \oplus x$, with $a \in \{0, 1\}$ and $y \rightarrow y \oplus 1$; here, \oplus denotes addition modulo 2.

Suppose Bob performs unknown measurements with POVM elements $M_{b|y}$ on his share of a bipartite quantum state $\rho^{AB} \in \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_d)$. The unnormalized conditional states on Alice's side are given by $\sigma_{b|y}^A = \text{Tr}(\mathbb{1} \otimes M_{b|y} \rho^{AB})$. Alice can do state tomography to determine these conditional states. The set of unnormalized conditional states is called an assemblage [32]. The above scenario exhibits steering if the state assemblage does not have a LHS model,

$$\sigma_{b|y}^A = \sum_{\lambda} p_{\lambda} P_{\lambda}(b|y) \rho_{\lambda}, \quad (6)$$

where $P_{\lambda}(b|y)$ are some conditional probability distributions and ρ_{λ} are positive operators which satisfy $\sum_{\lambda} \text{Tr } p_{\lambda} \rho_{\lambda} = 1$. The violation of a steering inequality as in Eq. (5) implies that the assemblage does not have a decomposition as in Eq. (6) [32]. Note that the state assemblage arising from any separable state has a LHS model. This implies that when the 1SDI scenario does not demonstrate steering, there always exists a separable state which reproduces the given state assemblage [33].

Nonseparability. Nonseparability of quantum correlations arises as a failure of the quantum separable model. In this model, LHS description is assumed for both of the parties. When a quantum correlation exhibits nonseparability, it violates a LHS-LHS model,

$$\text{Tr}(\rho M_{a|x} \otimes M_{b|y}) = \sum_{\lambda} p_{\lambda} P(a|A_x, \rho_{\lambda}^A) P(b|B_y, \rho_{\lambda}^B), \quad (7)$$

where $P(a|A_x, \rho_{\lambda}^A)$ and $P(b|B_y, \rho_{\lambda}^B)$ are the distributions arising from quantum states ρ_{λ}^A and ρ_{λ}^B , respectively. Any condition that is derived under the assumption of the above model is known as separability criterion or entanglement criterion.

A. Entanglement certification from CHSH and BB84 families

Moroder and Gittsovich (MG) [34] explored the task of entanglement detection in various partially DI scenarios. For instance, MG introduced entanglement certification in the semi-DI scenario where only the Hilbert-space dimensions of the subsystems are assumed. In Ref. [35], Goh *et al.* have defined a quantity which gives certifiable lower bounds on the amount of entanglement in the semi-DI scenario. By using this quantity, Goh *et al.* studied the amount of two-qubit entanglement certifiable from the CHSH family defined as

$$P_{CHSH} = \frac{2 + ab(-1)^{xy} \sqrt{2} V}{8}, \quad (8)$$

and the BB84 family defined as

$$P_{BB84} = \frac{1 + ab\delta_{xy}V}{4}. \quad (9)$$

The CHSH family with $V = 1$ violates the Bell-CHSH inequality to its quantum bound of $2\sqrt{2}$, whereas the BB84 family with $V = 1$ corresponds to the BB84 correlations [5]. The CHSH and BB84 families can be obtained from the two-qubit Werner state, $\rho_W = V|\Psi^-\rangle\langle\Psi^-| + (1-V)\mathbb{1}/4$, where $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, for suitable projective measurements. The CHSH family is achievable with the measurement settings that give rise to the maximal violation of the CHSH inequality in Eq. (3), whereas the BB84 family is achievable with the settings that give rise to the maximal violation of the steering inequality in Eq. (5).

As the CHSH family violates the Bell-CHSH inequality for $V > 1/\sqrt{2}$, it certifies entanglement in a DI way in this range. Since the BB84 family is local, it can also be produced by a separable state in the higher-dimensional space [5]. However, entanglement is certifiable from the BB84 family for $V > 1/\sqrt{2}$ in a 1SDI way, as it violates the steering inequality in this range.

Note that the two-qubit Werner state is entangled iff $V > 1/3$ [36]. If one assumes qubit dimension and which measurements are performed on both sides, the CHSH and BB84 families detect entanglement for $V > 1/2$. This follows from the fact that these correlations violate a quantum separable model in this range. This can be checked by the criteria to detect the nonexistence of a LHS-LHS model derived in Ref. [37]. The CHSH family with $V > 1/2$ violates the following LHS-LHS condition:

$$\langle A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 \rangle_{2 \times 2}^{LHS} \leq \sqrt{2}, \quad (10)$$

with the assumption that Alice and Bob have access to measurements that give rise to the optimal violation of the Bell-CHSH inequality in Eq. (3). The BB84 family with $V > 1/2$ violates the following LHS-LHS condition:

$$\langle A_0B_0 - A_1B_1 \rangle_{2 \times 2}^{LHS} \leq 1, \quad (11)$$

with the assumption that Alice and Bob have access to measurements that give rise to the optimal violation of the steering inequality in Eq. (5). Goh *et. al.* found that for $V > 1/2$, entanglement is certifiable from these two families if one assumes only qubit systems for Alice and Bob [35].

B. Incompatibility vs nonseparability

We now note that incompatibility of measurements is not necessary to demonstrate nonseparability. For this, we consider a measurement scenario in which Alice has access to the set of two dichotomic qubit POVMs, $\mathcal{M}_A^\eta = \{M_{\pm|\hat{a}_x}^\eta | x = 0, 1\}$, with elements

$$M_{\pm|\hat{a}_x}^\eta = \eta \Pi_{\pm|\hat{a}_x} + (1-\eta) \frac{\mathbb{1}}{2}; \quad 0 \leq \eta \leq 1, \quad (12)$$

and Bob has access to the set of two projective qubit measurements $\mathcal{M}_B = \{\Pi_{\pm|\hat{b}_y} | y = 0, 1\}$. Here $\Pi_{\pm|\hat{a}_x} = 1/2(\mathbb{1} \pm \hat{a}_x \cdot \vec{\sigma})$ and $\Pi_{\pm|\hat{b}_y} = 1/2(\mathbb{1} \pm \hat{b}_y \cdot \vec{\sigma})$ are projectors along the directions \hat{a}_x and \hat{b}_y ; $\vec{\sigma}$ is the vector of Pauli matrices. Note that in the above measurement scenario, Alice performs the noisy projective measurements. This implies that the correlation arising from the given two-qubit state ρ satisfies the following relation:

$$\text{Tr}(\rho M_{\pm|\hat{a}_x}^\eta \otimes \Pi_{\pm|\hat{b}_y}) = \text{Tr}(\rho_\eta \Pi_{\pm|\hat{a}_x} \otimes \Pi_{\pm|\hat{b}_y}). \quad (13)$$

Here, $\rho_\eta = \eta\rho + (1-\eta)\mathbb{1}/2 \otimes \rho_B$, with $\rho_B = \text{Tr}_A \rho$. In other words, the correlation arising from the given state for noisy projective measurements on Alice's side and projective measurements on Bob's side is equivalent to the correlation arising from the noisy state for the projective measurements on both sides. In Refs. [14, 38], this connection between the noisy measurements and noisy states has been used to obtain new results for the former from known results of the latter.

The set of two POVMs \mathcal{M}_A^η with $\hat{a}_0 \cdot \hat{a}_1 = 0$ is jointly measurable iff $\eta \leq 1/\sqrt{2}$ [39, 40] and noncommuting for any $\eta > 0$. We now illustrate that this noncommuting set of POVMs can be used to demonstrate nonseparability for $\eta > 1/2$.

Example 1. Suppose Alice and Bob share the singlet state, $|\Psi^-\rangle$, with Alice performing two noisy projective measurements along the directions $\hat{a}_0 = \hat{x}$ and $\hat{a}_1 = \hat{y}$ and Bob performing two projective measurements along the directions $\hat{b}_0 = -(\hat{x} + \hat{y})/\sqrt{2}$ and $\hat{b}_1 = (-\hat{x} + \hat{y})/\sqrt{2}$. From the relation given in Eq. (13), it follows that the statistics arising from this setting are analogous to that arising from the Werner state with the visibility η for projective measurements on both sides along the directions given above. Therefore, the statistics arising from the above scenario are equivalent to the CHSH family in Eq. (8) with V replaced by η .

Since the above statistics exhibit nonseparability for $\eta > 1/2$, the set of two POVMs which is jointly measurable in this range is also useful for this nonclassical task. This, in turn, implies that this compatible set of POVMs can also be used to detect two-qubit entanglement. For the measurements given in example 1, the statistics arising from the Werner state are equivalent to the CHSH family in Eq. (8) with V replaced by ηV . Thus, these statistics detect entanglement for any $\eta V > 1/2$. For $\eta = 1/\sqrt{2}$, the statistics do not exhibit Bell nonlocality; however, entanglement is detected for $V > 1/\sqrt{2}$.

III. TRIPARTITE NONLOCALITY

We now turn to the tripartite case which is the focus of this work. We restrict ourselves to the simplest scenario in which three spatially separated parties, i.e., Alice, Bob, and Charlie, perform two dichotomic measurements on their subsystems. The correlation is described by the conditional probability distributions: $P(abc|A_x B_y C_z)$, where $x, y, z \in \{0, 1\}$ and

$a, b, c \in \{-1, +1\}$. The correlation exhibits standard nonlocality (i.e., Bell nonlocality) if it cannot be explained by the LHV model,

$$P(abc|A_x B_y C_z) = \sum_{\lambda} p_{\lambda} P_{\lambda}(a|A_x) P_{\lambda}(b|B_y) P_{\lambda}(c|C_z). \quad (14)$$

If a correlation cannot be reproduced by this fully LHV model, it does not necessarily imply that it exhibits genuine nonlocality [16, 19].

In Ref. [16], Svetlichny introduced the strongest form of genuine tripartite nonlocality (see Ref. [19] for the other two forms of genuine nonlocality). A correlation exhibits Svetlichny nonlocality if it cannot be explained by a hybrid nonlocal-LHV (NLHV) model,

$$P(abc|A_x B_y C_z) = \sum_{\lambda} p_{\lambda} P_{\lambda}(a|A_x) P_{\lambda}(bc|B_y C_z) + \sum_{\lambda} q_{\lambda} P_{\lambda}(ac|A_x C_z) P_{\lambda}(b|B_y) + \sum_{\lambda} r_{\lambda} P_{\lambda}(ab|A_x B_y) P_{\lambda}(c|C_z), \quad (15)$$

with $\sum_{\lambda} p_{\lambda} + \sum_{\lambda} q_{\lambda} + \sum_{\lambda} r_{\lambda} = 1$. The bipartite probability distributions in this decomposition can have arbitrary nonlocality.

Svetlichny derived Bell-type inequalities to detect the strongest form of genuine nonlocality [16]. For instance, one of the Svetlichny inequalities reads,

$$\langle A_0 B_0 C_1 + A_0 B_1 C_0 + A_1 B_0 C_0 - A_1 B_1 C_1 \rangle + \langle A_0 B_1 C_1 + A_1 B_0 C_1 + A_1 B_1 C_0 - A_0 B_0 C_0 \rangle \leq 4. \quad (16)$$

Here $\langle A_x B_y C_z \rangle = \sum_{abc} abc P(abc|A_x B_y C_z)$. Quantum correlations violate the Svetlichny inequality (SI) up to $4\sqrt{2}$. A Greenberger-Horne-Zeilinger (GHZ) state [41] gives rise to the maximal violation of the SI for a different choice of measurements which do not demonstrate the GHZ paradox [42].

Bancal *et al.* [43] presented an intuitive approach to the SI in Eq. (16). For this, the SI was rewritten as follows:

$$\langle CHSH_{AB} C_1 + CHSH'_{AB} C_0 \rangle_{NLHV} \leq 4. \quad (17)$$

Here, $CHSH_{AB}$ is the canonical CHSH operator given in Eq. (3) and $CHSH'_{AB} = -A_0 B_0 + A_0 B_1 + A_1 B_0 + A_1 B_1$ is one of its equivalents. Bancal *et al.* observed that the input setting of Charlie defines which version of CHSH game Alice and Bob are playing. When C gets the input $z = 0$, AB play the canonical $CHSH$ game; when C gets the input $z = 1$, AB play $CHSH'$. In Argument 1 of Ref. [43], Bancal *et al.* found that AB play the average game $\pm CHSH \pm CHSH'$, where the signs indicate that which game they are playing depends on the outputs of C . It can be checked that the algebraic maximum of any of these average games is 4 for the NLHV model in Eq. (15) with $\sum_{\lambda} r_{\lambda} = 1$.

A. Svetlichny steering

We will derive a criterion for tripartite EPR-steering by exploiting the structure of the SI given in Eq. (17). For this,

we consider the following 1SDI scenario. Alice and Bob have access to incompatible qubit measurements that give rise to violation of a Bell-CHSH inequality, while Charlie has access to two black-box measurements. Suppose that $P(abc|A_x B_y C_z)$ cannot be explained by the following nonlocal LHS-LHV (NLHS) model:

$$P(abc|A_x B_y C_z) = \sum_{\lambda} p_{\lambda} P(ab|A_x B_y, \rho_{AB}^{\lambda}) P_{\lambda}(c|C_z) + \sum_{\lambda} q_{\lambda} P(a|A_x, \rho_A^{\lambda}) P(b|B_y, \rho_B^{\lambda}) P_{\lambda}(c|C_z), \quad (18)$$

where $P(ab|A_x B_y, \rho_{\lambda})$ denotes the nonlocal probability distribution arising from two-qubit state ρ_{AB}^{λ} , and $P(a|A_x, \rho_A^{\lambda})$ and $P(b|B_y, \rho_B^{\lambda})$ are the distributions arising from qubit states ρ_A^{λ} and ρ_B^{λ} . Then the quantum correlation exhibits genuine steering from Charlie to Alice and Bob.

We obtain the following criterion for genuine steering under the constraint of the above 1SDI scenario.

Theorem 1. *If a given quantum correlation violates the steering inequality*

$$\langle CHSH_{AB} C_1 + CHSH'_{AB} C_0 \rangle_{2 \times 2 \times ?}^{NLHS} \leq 2\sqrt{2}, \quad (19)$$

then it exhibits genuine tripartite steering from Charlie to Alice and Bob. Here, $2 \times 2 \times ?$ indicates that Alice and Bob have access to known qubit measurements that demonstrate Bell nonlocality, while Charlie's measurements are uncharacterized.

Proof. Note that in the 1SDI scenario that we are interested in, AB play the average game $\pm CHSH \pm CHSH'$. The maximum of any of these games cannot exceed $2\sqrt{2}$ if the correlation admits the NLHS model given in Eq. (18). There are two cases which have to be checked: (i) Suppose Alice and Bob have a LHS-LHS model, the expectation values of the CHSH operators are bounded by $-\sqrt{2} \leq \langle CHSH \rangle_{2 \times 2}^{LHS} \leq \sqrt{2}$ and $-\sqrt{2} \leq \langle CHSH' \rangle_{2 \times 2}^{LHS} \leq \sqrt{2}$ [37]. This implies that $\langle \pm CHSH \pm CHSH' \rangle_{2 \times 2}^{LHS} \leq 2\sqrt{2}$. (ii) In case Bell nonlocality is shared by Alice and Bob, it can be checked that $\langle \pm CHSH \pm CHSH' \rangle_{2 \times 2} \leq 2\sqrt{2}$. For instance, the quantum correlation which exhibits maximal Bell nonlocality has $\langle CHSH \rangle = 2\sqrt{2}$ and $\langle CHSH' \rangle = 0$. Thus, the violation of the inequality in Eq. (19) implies the violation of the model in Eq. (18). \square

For a given quantum state, genuine steering as witnessed by the steering inequality in Eq. (19) originates from measurement settings that give rise to Svetlichny nonlocality. For this reason, we call this type of genuine steering Svetlichny steering.

Definition 1. *A quantum correlation exhibits Svetlichny steering if it cannot be explained by a model in which arbitrary two-qubit Bell nonlocality is allowed between two parties with the third party locally correlated.*

Note that the SI is invariant under the permutation of the parties. This implies that the criterion to detect Svetlichny

steering from A to BC or B to AC can be obtained from the steering inequality in Eq. (19) by permuting the parties.

We consider the Svetlichny family defined as

$$P_{SV} = \frac{2 + abc(-1)^{xy \oplus xz \oplus yz \oplus x \oplus y \oplus z \oplus 1} \sqrt{2}V}{16}, \quad (20)$$

which is the tripartite version of the CHSH family in Eq. (8).

Example 2. The Svetlichny family can be obtained from a noisy three-qubit GHZ state, $\rho = V|\Phi_{GHZ}\rangle\langle\Phi_{GHZ}| + (1-V)\mathbb{I}/8$, where $|\Phi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, for the measurements that give rise to the maximal violation of the SI; for instance, $A_0 = \sigma_x$, $A_1 = \sigma_y$, $B_0 = (\sigma_x - \sigma_y)/\sqrt{2}$, $B_1 = (\sigma_x + \sigma_y)/\sqrt{2}$, $C_0 = \sigma_x$, and $C_1 = -\sigma_y$.

Note that the noisy GHZ state given above is genuinely entangled iff $V > 0.429$ [44]. The Svetlichny family certifies genuine entanglement in a DI way for $V > 1/\sqrt{2}$, as it violates the SI in this range.

The Svetlichny family can be written as a convex mixture of the local deterministic strategies when $V \leq 1/\sqrt{2}$ [19]. This implies that in this range, it can also arise from a separable state in the higher dimensional space [5]. However, the measurement statistics in example 2 certify genuine entanglement for $V > 1/2$ in a 1SDI way, as these statistics violate the steering inequality in Eq. (19).

By using the concept of steering, Bancal *et al.* [43] observed that the structure of the SI in Eq. (17) allows one to understand its violation by genuinely entangled states: The SI should be violated iff Charlie's measurements prepare entangled states for Alice and Bob such that the average game $\pm CHSH \pm CHSH' > 4$. Similarly, we understand violation of the steering inequality by the Svetlichny family as follows. When the parties share the noisy GHZ state and observe the optimal violation of the steering inequality in Eq. (19), we have the following two situations. First, Charlie's black-box measurements prepare the noisy Bell states, which are a mixture of a Bell state (i.e., a maximally entangled state) and white noise, with the visibility V for Alice and Bob. Second, Alice and Bob's measurements generate the CHSH family from the states steered by Charlie. These imply that the Svetlichny family certifies genuine tripartite entanglement for $V > 1/2$ if one assumes only qubit dimension for two parties.

B. Mermin steering

We now derive a criterion for another form of tripartite EPR-steering from a Mermin inequality (MI) [45],

$$\langle A_0 B_0 C_1 + A_0 B_1 C_0 + A_1 B_0 C_0 - A_1 B_1 C_1 \rangle_{LHV} \leq 2. \quad (21)$$

In the seminal paper [45], the MI was derived to demonstrate standard nonlocality of three-qubit correlations arising from the genuinely entangled states. For this purpose, noncommuting measurements that do not demonstrate Svetlichny nonlocality were used. Note that when the GHZ state maximally violates the MI, the measurements that give rise to it exhibit the GHZ paradox [42].

We rewrite the MI in Eq. (21) as follows:

$$\langle Mermin_{AB} C_1 + Mermin'_{AB} C_0 \rangle_{LHV} \leq 2, \quad (22)$$

where $Mermin_{AB} = A_0 B_0 - A_1 B_1$ and $Mermin'_{AB} = A_0 B_1 + A_1 B_0$ ¹. Note that these bipartite Mermin operators can be used to witness EPR-steering without Bell nonlocality as in Eq. (5). Inspired by the structure of the MI in Eq. (22), we consider the following 1SDI scenario. Alice and Bob have access to incompatible qubit measurements that give rise to EPR-steering without Bell nonlocality, while Charlie has access to two dichotomic black-box measurements. Suppose that $P(abc|A_x B_y C_z)$ cannot be explained by the following steering LHS-LHV (SLHS) model:

$$P(abc|A_x B_y C_z) = \sum_{\lambda} p_{\lambda} P_L^{EPR}(ab|A_x B_y, \rho_{AB}^{\lambda}) P_{\lambda}(c|C_z) + \sum_{\lambda} q_{\lambda} P(a|A_x, \rho_A^{\lambda}) P(b|B_y, \rho_B^{\lambda}) P_{\lambda}(c|C_z), \quad (23)$$

where $P_L^{EPR}(ab|A_x B_y, \rho_{AB}^{\lambda})$ denotes the local probability distribution which exhibits EPR-steering. Then the quantum correlation exhibits genuine steering from Charlie to Alice and Bob.

We obtain the following criterion which witnesses genuine steering in the above 1SDI scenario.

Theorem 2. If a given quantum correlation violates the steering inequality

$$\langle Mermin_{AB} C_1 + Mermin'_{AB} C_0 \rangle_{2 \times 2 \times ?}^{SLHS} \leq 2, \quad (24)$$

then it exhibits genuine tripartite steering from Charlie to Alice and Bob. Here Alice and Bob's measurements demonstrate EPR-steering without Bell nonlocality, while Charlie's measurements are uncharacterized.

Proof. Notice that the correlations that admit the SLHS model in Eq. (23) also admit the standard LHV model in Eq. (14). This implies that if a given correlation violates the inequality in Eq. (24), the correlation also violates the SLHS model.

Similar to the proof of the inequality in Eq. (19), we will also show that the inequality in Eq. (24) is satisfied if the correlation admits the SLHS model. Notice that in the 1SDI scenario that we are now interested in, AB play the average game $\pm Mermin \pm Mermin'$. The algebraic maximum of any of these games is 2 if the correlation admits the SLHS model. There are two cases that have to be checked now. (i) Suppose Alice and Bob have a LHS-LHS model, the expectation values of the Mermin operators satisfy $-1 \leq \langle Mermin \rangle_{2 \times 2}^{LHS} \leq 1$ and $-1 \leq \langle Mermin' \rangle_{2 \times 2}^{LHS} \leq 1$ [37]. This implies that $\langle \pm Mermin \pm Mermin' \rangle_{2 \times 2}^{LHS} \leq 2$. (ii) In case EPR-steering is shared by Alice and Bob, it can be checked that $\langle \pm Mermin \pm Mermin' \rangle \leq 2$. For instance, the quantum correlation which violates the steering inequality in Eq. (5) maximally has $\langle Mermin \rangle = 2$ and $\langle Mermin' \rangle = 0$. \square

¹ The multipartite generalization of these operators generates the Mermin inequalities, hence the name [45, 46].

For a given state, genuine steering as witnessed by the steering inequality in Eq. (24) originates from measurement settings that lead to standard nonlocality, which is witnessed only by the violation of the MI. For this reason, we call this type of tripartite steering Mermin steering.

Definition 2. A quantum correlation exhibits Mermin steering if it cannot be explained by a model in which arbitrary two-qubit EPR-steering without Bell nonlocality is allowed between two parties with the third party locally correlated.

Note that the MI is invariant under the permutation of the parties. This implies that the criterion to detect Mermin steering from A to BC or B to AC can be obtained from the steering inequality in Eq. (24) by permuting the parties.

We consider the GHZ family defined as

$$P_{GHZ} = \frac{1 + abc\delta_{x\oplus y, z\oplus 1}V}{8}, \quad (25)$$

which is the tripartite version of the BB84 family in Eq. (9).

Example 3. The GHZ family can be obtained from the noisy three-qubit GHZ state for the measurements that give rise to the GHZ paradox; for instance, $A_0 = \sigma_x$, $A_1 = \sigma_y$, $B_0 = \sigma_x$, $B_1 = \sigma_y$, $C_0 = \sigma_x$, and $C_1 = -\sigma_y$.

Since the GHZ family does not exhibit genuine nonlocality, it does not certify genuine entanglement in a DI way. However, genuine entanglement is detected from the measurement statistics in example 3 for $V > 1/2$ in a 1SDI way, as these statistics violate the steering inequality in Eq. (24). Notice that when the GHZ family violates this steering inequality, Alice and Bob's measurement settings generate the BB84 family from the noisy Bell states with the visibility V which are steered by Charlie. This implies that genuine entanglement is certifiable from the GHZ family for $V > 1/2$ by assuming only qubit systems for two parties.

IV. INCOMPATIBILITY VERSUS TRIPARTITE STEERING

Following the approach of Ref. [27], we will now discuss tripartite steerability in the 1SDI scenarios (see Fig. 1) which we have considered. Let $\rho^{ABC} \in \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_d)$ denote the shared tripartite quantum state and $M_{c|z}$ denote the measurement operators on Charlie's side. The assemblage, i.e., the set of (unnormalized) conditional states on Alice and Bob's side, is given by

$$\sigma_{c|z}^{AB} = \text{Tr}_C(\mathbb{1} \otimes \mathbb{1} \otimes M_{c|z} \rho^{ABC}). \quad (26)$$

Note that in examples 2 and 3, there are genuinely tripartite entangled states which do not demonstrate tripartite steering. Suppose the 1SDI scenario which uses genuine tripartite entanglement does not demonstrate tripartite steering. The biseparable state which can reproduce the given assemblage can be written as

$$\begin{aligned} \rho_{bisep}^{ABC} = & \sum_{\lambda} p_{\lambda}^{A:BC} \rho_{\lambda}^A \otimes \rho_{\lambda}^{BC} + \sum_{\mu} p_{\mu}^{B:AC} \rho_{\mu}^B \otimes \rho_{\mu}^{AC} \\ & + \sum_{\nu} p_{\nu}^{AB:C} \rho_{\nu}^{AB} \otimes \rho_{\nu}^C. \end{aligned} \quad (27)$$

Thus, the unsteerable assemblage admits the following LHS model:

$$\begin{aligned} \sigma_{c|z}^{AB} = & \sum_{\lambda} p_{\lambda}^{A:BC} \rho_{\lambda}^A \otimes \sigma_{c|z,\lambda}^B + \sum_{\mu} p_{\mu}^{B:AC} \sigma_{c|z,\mu}^A \otimes \rho_{\mu}^B \\ & + \sum_{\nu} p_{\nu}^{AB:C} P_{\nu}(c|z) \rho_{\nu}^{AB}, \end{aligned} \quad (28)$$

where $\sigma_{c|z,\lambda}^B = \text{Tr}_C(\mathbb{1} \otimes M'_{c|z} \rho_{\lambda}^{BC})$, $\sigma_{c|z,\mu}^A = \text{Tr}_C(\mathbb{1} \otimes M'_{c|z} \rho_{\mu}^{AC})$ and $P_{\nu}(c|z) = \text{Tr}(M'_{c|z} \rho_{\nu}^C)$.

Suppose Charlie has access to POVMs which are compatible. Then there exists a parent POVM with measurement operators G_{ν} such that

$$M_{c|z} = \sum_{\nu} D_{\nu}(c|z) G_{\nu}, \quad (29)$$

where $D_{\nu}(c|z)$ are positive numbers with $\sum_{\nu} D_{\nu}(c|z) = 1$ [47]. For these measurements, the state assemblage of Alice and Bob's side admits the following decomposition:

$$\sigma_{c|z}^{AB} = \sum_{\nu} D_{\nu}(c|z) \text{Tr}_C(\mathbb{1} \otimes \mathbb{1} \otimes G_{\nu} \rho^{ABC}). \quad (30)$$

Note that the above decomposition resembles that of the unsteerable assemblage given in Eq. (28). Therefore, if Charlie's measurement settings are compatible then there is no steering between Charlie and Alice-Bob.

Inspired by example 1, we consider the following measurement scenario.

Example 4. Alice and Bob perform projective measurements along the directions $\hat{a}_0 = \hat{x}$, $\hat{a}_1 = \hat{y}$, $\hat{b}_0 = (\hat{x} - \hat{y})/\sqrt{2}$ and $\hat{b}_1 = (\hat{x} + \hat{y})/\sqrt{2}$. Charlie performs noisy projective measurements with visibility η as in Eq. (12) along the directions $\hat{c}_0 = \hat{x}$ and $\hat{c}_1 = -\hat{y}$. For the above measurements, the statistics arising from the GHZ state, $|\Phi_{GHZ}\rangle$, are equivalent to the Svetlichny family in Eq. (20) with V replaced by η .

Note that in the above example, Charlie's measurement settings are compatible for $\eta \leq 1/\sqrt{2}$. This implies that in this range, the state assemblage of Alice and Bob's side has the decomposition as in Eq. (30). Therefore, Charlie does not demonstrate tripartite steerability to Alice and Bob for $\eta \leq 1/\sqrt{2}$. However, the correlations in example 4 violate the steering inequality in Eq. (19) for any $\eta > 1/2$. Note that in this example, Bob and Charlie's measurement settings can be used to demonstrate Bell nonlocality for $\eta > 1/\sqrt{2}$ and nonseparability for $\eta > 1/2$ (as in example 1). Therefore, the correlations in example 4 exhibit Svetlichny steering from A to BC for $\eta > 1/2$.

Remark 1. Quantum correlations can also detect tripartite EPR-steering from Alice to Bob and Charlie even if Charlie's measurement settings are compatible.

V. CONCLUSION

We have introduced Svetlichny steering and Mermin steering to distinguish the presence of two types of genuine tripartite steering. We have derived the two inequalities which

detect them by using the structure of the Svetlichny inequality and Mermin inequality, which detect genuine nonlocality and standard nonlocality, respectively. We find that genuine tripartite entanglement is certifiable from the correlations that violate one of these steering inequalities in a semi-device-independent way as well. That is, when qubit dimension is assumed for two parties, the correlations which exhibit Svetlichny steering or Mermin steering also detect genuine tripartite entanglement (where measurements on all subsystems are uncharacterized). We demonstrate that quantum correlations can also detect tripartite EPR-steering from Alice to Bob and Charlie even if Charlie's measurement settings are compatible. The two tripartite steering inequalities which detect Svetlichny steering and Mermin steering can be generalized to arbitrary number of systems by using the structure of n -partite Svetlichny inequality [17, 18] and Mermin-Ardehali-

Belinskii-Klyshko (MABK) inequality [45, 48–51], respectively.

ACKNOWLEDGEMENT

I would like to thank Dr. J.-D. Bancal and Prof. H. M. Wiseman for the helpful conversations on steering. I wish to thank Dr. K. P. Yogendran and Prof. Suresh Govindarajan for useful comments. I am thankful to Dr. Manik Banik for fruitful discussions. I am very thankful to an anonymous referee for the helpful comments and suggestions to improve the quality of the manuscript. This work was partially funded by Department of Science and Technology, via INSPIRE Project No. PHY1415305DSTXPRAN, hosted at IIT Madras.

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